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DESIGN PROBLEM OF LAMINATED PLATES WITH SPECIFIED CHARACTERISTICS

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The problem of design of laminated plates with specified stiffness and strength characteristics is considered. The starting design problem is reduced to the convex-combination problems, which is solved by the convolution method. The following design problems are solved: design with allowance for strength, design of a laminated plate of unconstrained thickness, and design for approximately specified characteristics.

Key words: *design problem, laminated plate, stiffness, strength.*

Many studies have dealt with the problem of design of laminated plates. The following problem of optimal design has been extensively studied: Obtain a design of a plate that minimizes a certain functional (weight, deflection, etc.) [1, 2]. Less attention has been given to the design problem formulated as follows [3–5]: Find the method of design of a plate with specified (not necessarily optimal) characteristics. In the present paper, we show that strength restrictions can be taken into account in the existing algorithms of design of laminated plates with specified stiffness.

1. Formulation of the Problem. We consider a laminated plate composed of homogeneous isotropic layers parallel to the coordinate plane (see Fig. 1). We denote the coordinate in the transverse direction of the plate by y. For the laminated plate, Young's modulus $E(y)$ and Poisson's ratio $\nu(y)$ are functions of y.

It is required to find the material distribution in the layers such that the plate has specified stiffnesses (stiffness in the S^0_{ijk} plane, nonsymmetric stiffnesses S^1_{ijkl}, and flexural stiffnesses S^2_{ijkl}). To this end, it is necessary to solve the system of integral equations of the first kind

$$h^{\mu+1}\int\limits_{-1/2}^{1/2} E(y)\,\frac{y^\mu}{1-\nu^2(y)}\,dy = S^\mu_{iiii}, \qquad \mu = 0,1,2, \quad i = 1,$$

$$h^{\mu+1}\int\limits_{-1/2}^{1/2} E(y)\,\frac{y^\mu}{1+\nu(y)}\,dy = S^\mu_{1212}, \qquad h^{\mu+1}\int\limits_{-1/2}^{1/2} E(y)\nu(y)\,\frac{y^\mu}{1-\nu^2(y)}\,dy = S^\mu_{1122} \tag{1.1}$$

for the functions $E(y)$ and $\nu(y)$ for given parameters S^μ_{iiii}, S^μ_{1212}, and S^μ_{1122}.

The integrals in (1.1) represent stiffnesses of the plate expressed in terms of the elastic constants of the layers [2, 4]. System (1.1) is written for the plate of unit thickness (the general case is considered below).

For simplicity, we set $\nu(y) = \text{const}$ [the case of $\nu(y) \neq \text{const}$ is considered below]. In this case, problem (1.1) is a problem for one function $E(y)$.

The plates are made of a finite number of materials. Hence, the function $E(y)$ takes a finite number of values.

We divide the plate into m layers of equal thickness $\delta = 1/m$. The function $E(y)$ is constant within the intervals $[-1/2 + (i-1)/m, -1/2 + i/m)$. Here and below, $i = 1, \ldots, m$.

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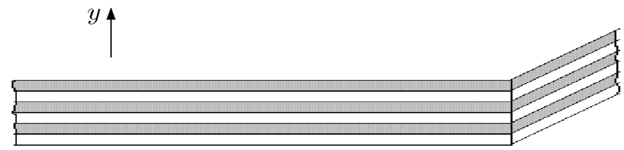

Fig. 1

We introduce the notation $d_{1i} = \dfrac{1}{\delta} \displaystyle\int\limits_{-1/2+(i-1)/m}^{1/2+i/m} y\,dy$ and $d_{2i} = \dfrac{1}{\delta} \displaystyle\int\limits_{-1/2+(i-1)/m}^{1/2+i/m} y^2\,dy$. System (1.1) becomes

$$\sum_{i=1}^{m} E_i \delta = S^0, \qquad \sum_{i=1}^{m} E_i d_{1i} \delta = S^1, \qquad \sum_{i=1}^{m} E_i d_{2i} \delta = S^2. \tag{1.2}$$

The quantities S^0, S^1, and S^2 are expressed in terms of the right sides of Eqs. (1.1) and ν. Dividing equalities (1.2) by S^0, we obtain the problem

$$\sum_{i=1}^{m} x_i = 1, \quad x_i \in Z_n, \qquad \sum_{i=1}^{m} x_i \boldsymbol{v}_i = \boldsymbol{v}, \tag{1.3}$$

where $x_i = E_i \delta / S^0$, $\boldsymbol{v}_i = (d_{1i}, d_{2i})$ $(i = 1, \ldots, m)$, and $\boldsymbol{v} = (S^1/S^0, S^2/S^0)$. The unknowns in problem (1.3) are E_i. The physical meaning implies that $E_i \geqslant 0$.

2. Discrete Convex-Combination Problem. We consider the following problem. Let $Z_n \subset [0, 1]$ be a finite set (consisting of n numbers) and $\boldsymbol{v}_i$ ($\boldsymbol{v} \in R^k$) be specified vectors. It is required to find the numbers x_i that are the solution of the problem

$$\sum_{i=1}^{m} \boldsymbol{v}_i x_i = \boldsymbol{v}; \tag{2.1}$$

$$\sum_{i=1}^{m} x_i = 1; \tag{2.2}$$

$$x_i \in Z_n, \quad i = 1, \ldots, m. \tag{2.3}$$

Problem (2.1), (2.2) subject to

$$0 \leqslant x_i \leqslant 1 \tag{2.4}$$

is the convex-combination problem (CCP) considered in [5]. Problem (2.1)–(2.3) is a discrete CCP.

The general solution of problem (2.1)–(2.3) (i.e., set of all its solutions) can be constructed in the following manner. We replace the discreteness condition (2.3) by condition (2.4). The general solution of the CCP (2.1), (2.2), (2.4) obtained in [5] (see also [4, 6]) has the form

$$x_i = \sum_{\gamma=1}^{M} P_{i\gamma} \lambda_\gamma \qquad (i = 1, \ldots, m), \tag{2.5}$$

where $M < \infty$ and λ_γ $(\gamma = 1, \ldots, M)$ are any numbers satisfying the conditions

$$\sum_{\gamma=1}^{M} \lambda_\gamma = 1, \qquad 0 \leqslant \lambda_\gamma \leqslant 1. \tag{2.6}$$

The vectors $\boldsymbol{P}_\gamma = (P_{1\gamma}, \ldots, P_{m\gamma})$ $(\gamma = 1, \ldots, M)$ are the so-called simplicial solutions of the CCP (2.1), (2.2), (2.4), constructed by the method proposed in [5]. For random perturbations of the coefficients $\{\boldsymbol{v}_i\}$ and free term $\boldsymbol{v}$ in the CCP (2.1), (2.2), (2.4), the system of vectors $\{\boldsymbol{P}_\gamma,\ \gamma = 1, \ldots, M\}$ coincides with the set of the end points of the polyhedron $\Lambda(\boldsymbol{v})$ with a probability of 1. Consequently, it is the minimum system of points generating the set $\Lambda(\boldsymbol{v})$ [7]. This property can be used to solve the problem numerically by the method proposed in [5].

The set $\Lambda(\boldsymbol{v})$ of the solutions of the CCP (2.1), (2.2), (2.4) can be written in the form $\Lambda(\boldsymbol{v}) = \operatorname{conv}\{\boldsymbol{P}_\gamma$ and $\gamma = 1, \ldots, M\}$ (conv denotes the convex combination [8]). The set $Z_n^m = \{\boldsymbol{x}\colon\ x_i \in Z_n\}$ is a discrete grid in R^m. The set of solutions of the CPP (2.1)–(2.3) is $\Lambda(\boldsymbol{v}) \cap Z_n^m$. Problem (2.1)–(2.3) is solved once the quantities satisfying the condition $x_i \in Z_n$ are found among the quantities x_i determined by formulas (2.5) and (2.6).

Relations (2.5) and (2.6) can be considered as the CCP for the quantities λ_γ. The convexity of $\Lambda(\boldsymbol{v})$ implies that, if the first $i-1$ equations in (2.5) are satisfied, the next ith equation is solvable if and only if

$$x_i \in I_i = [\min_i, \max_i]. \tag{2.7}$$

It should be noted that the interval I_i depends on the choice of $x_1, \ldots, x_{i-1}$.

From (2.7), we obtain the necessary and sufficient condition for the existence of the solution of the discrete CCP $Z(i) = Z_n \cap I_i \neq \varnothing$ for all $i = 1, \ldots, m$.

Since the intervals I_i depend on the choice of $x_1, \ldots, x_{i-1}$, the tree T arises. We denote the root of this tree by $T(0)$. Branching of the tree at the level $T(i-1)$ is determined by points $Z(i)$. Any branch that passes from the root $T(0)$ to the level $T(m)$ gives the solution of the discrete CCP (2.1)–(2.3). Conversely, any solution of the discrete CCP corresponds to the branch that passes from the root $T(0)$ to the level $T(m)$. Thus, once the tree T is constructed, we find the entire set of solutions of the discrete CCP [9].

We describe the step of the iterative algorithm of constructing the tree T. Let the design fragment $x_1, \ldots, x_{i-1}$ be available (i.e., the first $i-1$ layers of the plate are filled). At the ith step, all the available fragments $x_1, \ldots, x_{i-1}$ are supplemented by the quantities x_i that satisfy (2.7) for the corresponding fragments $x_1, \ldots, x_{i-1}$. As a result, we obtain the fragment $x_1, \ldots, x_{i-1}, x_i$.

The main procedures of the numerical algorithm are described in [9]. It follows from [9] that the solution of the discrete CCP reduces to the solution of the CCP and linear-programming problem.

3. Approximate Solutions of the CCP. The set $\Lambda(\boldsymbol{v})$ (2.1), (2.2), (2.4) belongs to a hyperplane and its dimension is smaller than m [10]. The points from Z_n^m located near the set $\Lambda(\boldsymbol{v})$ correspond to approximate designs (designs of plates whose stiffnesses are close to specified values). According to [10], to find approximate designs, one should perturb the set $\{\boldsymbol{P}_\gamma,\ \gamma = 1, \ldots, M\}$ by vectors $\boldsymbol{w}_i = (\boldsymbol{v}_i, 1)$ orthogonal to the set $\Lambda(\boldsymbol{v})$ and then solve the CCP for this system of vectors.

Let the perturbation of the set $\{\boldsymbol{P}_\gamma,\ \gamma = 1, \ldots, M\}$ be of the form $\{\boldsymbol{P}_{M+1+j} = \boldsymbol{P}_{j+1} + \zeta \boldsymbol{w}_j,\ j = 0, 1, 2\}$, where ζ is the characteristic magnitude of perturbation. Thus, three vectors with components orthogonal to $\Lambda(\boldsymbol{v})$ are added to the system of vectors $\{\boldsymbol{P}_\gamma,\ \gamma = 1, \ldots, M\}$.

4. Averaged Strength Criterion. The averaged strength criterion of a composite is understood as a criterion written in terms of characteristics of a plate as a two-dimensional object, which allows one to estimate the strength of this plate as a three-dimensional (though thin) heterogeneous body. These criteria can be obtained from the formulas relating the local stresses σ_{ij}^ε in the plate considered as a three-dimensional body to the strain characteristics of this plate as a two-dimensional object. For a laminated plate, these formulas were obtained, for example, in [4, 6] and have the form

$$\sigma_{ij}^\varepsilon = c_{ij\alpha\beta}(y)[\varepsilon_{\alpha\beta} + (y - S^1/S^0)\rho_{\alpha\beta}] \qquad (i, j = 1, 2, 3, \quad \alpha, \beta = 1, 2),$$

where $c_{ijkl}(y)$ is the elastic-constant tensor, $\varepsilon_{\alpha\beta}$ is the strain tensor in the plate plane, and $\rho_{\alpha\beta}$ is the curvature tensor.

Given the relations between $\varepsilon_{\alpha\beta}$ and $\rho_{\alpha\beta}$ and in-plane forces $N_{\alpha\beta}$ and moments $M_{\alpha\beta}$,

$$N_{\alpha\beta} = S^0_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} + S^1_{\alpha\beta\gamma\delta}\rho_{\gamma\delta}, \qquad M_{\alpha\beta} = S^1_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} + S^2_{\alpha\beta\gamma\delta}\rho_{\gamma\delta},$$

one can express σ_{ij}^ε in terms of $N_{\alpha\beta}$ and $M_{\alpha\beta}$.

We consider the case where Poisson's coefficients of the layers are equal. The elastic-constant tensor can be written as

$$c_{ijkl}(y) = E(y)c^0_{ijkl},$$

where c^0_{ijkl} is independent of y. Then, if the ith layer of the plate $[-1/2 + (i-1)/m, -1/2 + i/m]$ is filled by the Kth material, the local stresses in this layer are calculated by the formula

$$\sigma_{ij}^\varepsilon = E_K c^0_{ijkl}[\varepsilon_{\alpha\beta} + (y - S^1/S^0)\rho_{\alpha\beta}], \qquad y \in [-1/2 + (i-1)/m, -1/2 + i/m). \tag{4.1}$$

Let the strength criterion of the Kth material be of the form

$$f_K(\sigma^{\varepsilon}_{ij}) < 1, \tag{4.2}$$

where f_K is the nonnegative Lipschitz function.

Substituting (4.1) into (4.2), we obtain the averaged strength criterion for the ith layer under the condition that it is filled by the Kth material:

$$F_K(\varepsilon_{\alpha\beta}, \rho_{\alpha\beta}) \equiv f_K\{E_K c^0_{ijkl}[\varepsilon_{\alpha\beta} + (y - S^1/S^0)\rho_{\alpha\beta}]\} < 1. \tag{4.3}$$

The plate was divided into layers with a step $1/m$. In this case, (4.3) can be replaced by the condition

$$F_K(\varepsilon_{\alpha\beta}, \rho_{\alpha\beta}) \equiv f_K\{E_K c^0_{ijkl}[\varepsilon_{\alpha\beta} + (-1/2 + i/m - S^1/S^0)\rho_{\alpha\beta}]\} < 1 \tag{4.4}$$

with an error M/m (M is the maximum of the Lipschitz constants for the functions f_K, where $K = 1, \ldots, n$).

For the design $\{E_1, \ldots, E_m\}$ or $\{x_1, \ldots, x_m\}$, all layers of the plate remain undamaged provided that condition (4.4) holds for all $i = 1, \ldots, m$. If condition (4.3) is not satisfied, the corresponding layer fails. Conditions (4.3) and (4.4) are the exact and approximate strength conditions, respectively.

5. Design Problem with Allowance for Strength. It is required to find all designs of plates with specified stiffnesses S^μ ($\mu = 0, 1, 2$) which can sustain the strains $\varepsilon_{\alpha\beta}$ and $\rho_{\alpha\beta}$ (or loads $N_{\alpha\beta}$ and $M_{\alpha\beta}$) without failure of the layers. The mathematical formulation of the problem is as follows: Solve problem (2.1)–(2.3) subject to (4.4). To solve the problem, one can employ the algorithm for solving the CCP described in Sec. 2 using condition (4.4) as a filter at the current step of this algorithm. To substantiate this statement, we describe the step of the algorithm for solving the CCP (see Sec. 2). At the ith step, the existing design fragment $\{x_1, \ldots, x_{i-1}\}$ is supplemented by the quantity x_i that satisfies condition (2.7).

We change the step of the algorithm in the following manner. Let x_K satisfy (2.7) (the subscripts i and K refer to the layer and material numbers, respectively). We verify whether the following condition holds:

$$F_{Ki}(\varepsilon_{\alpha\beta}, \rho_{\alpha\beta}) \equiv f_K\{E_K c^0_{ijkl}[\varepsilon_{\alpha\beta} + (-1/2 + i/m)\rho_{\alpha\beta}]\} < 1. \tag{5.1}$$

If condition (5.1) is satisfied, $x_K = E_K/(mS^0)$ is taken as a possible value. Obviously, this corresponds to the algorithm described in Sec. 2, in which the condition $x_i \in I_i$ is replaced by

$$x_K \in I_i \cap \{x_R \colon x_R = E_K/(mS^0)\}, \tag{5.2}$$

where x_R satisfies condition (5.1). Thus, at the step of the algorithm described in Sec. 2, the additional condition (5.1), which we call strength-criterion filter, should hold.

Condition (5.2) has the mechanical meaning. If this condition is satisfied, the material with Young's modulus E_K (x_K and E_K are uniquely related) is a candidate for a filler of the ith layer.

The modification of the algorithm proposed allows one to obtain all solutions of the design problem (2.1)–(2.3) with condition (4.4). This follows from the fact that the set of vectors (2.5) yields all solutions of the CCP and any solution of the design problem (2.1)–(2.3), (4.4) satisfies (2.5) and (5.2).

In the case where the loads $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are specified, the strains $\varepsilon_{\alpha\beta}$ and $\rho_{\alpha\beta}$ can be expressed in terms of $N_{\alpha\beta}$ and $M_{\alpha\beta}$. As a result, we arrive at the problem considered above.

6. Introducing the Plate Thickness into the Set of Design Variables. We consider a plate of unconstrained thickness h. The stiffnesses $S^\mu_{iiii}(h)$ ($\mu = 0, 1, 2$) of the plate of unconstrained thickness can be written in the form $S^\mu_{iiii}(h) = h^{\mu+1} S^\mu_{iiii}$, where S^μ_{iiii} ($\mu = 0, 1, 2$) are the stiffnesses of the plate of unit thickness [see (1.1)]. Using this expression, we transform Eqs. (2.1)–(2.3) to the discrete CCP with the right side dependent on h:

$$\sum_{i=1}^{m} \boldsymbol{v}_i x_i = \Big(\frac{v_1}{h}, \frac{v_2}{h^2}\Big), \qquad \sum_{i=1}^{m} x_i = 1. \tag{6.1}$$

[x_i satisfy condition (2.3)]. In (6.1), the vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}$ are the same as in Sec. 1.

To design a plate of thickness h with the stiffnesses S^μ_{1111} ($\mu = 0, 1, 2$), it is necessary to solve problem (6.1) for a specified value of h. If the plate thickness is not specified, one can vary it in steps δh within a certain interval $[h_{\min}, h_{\max}]$ and solve problem (6.1) with the right side $(v_1/h, v_2/h^2)$, where $h = h_{\min} + p\delta h$, $p = 1, \ldots, \mathrm{int}((h_{\max} - h_{\min})/(\delta h))$ (int is the integer part).

The interval $[h_{\min}, h_{\max}]$ (if it is not specified) can be estimated for the given stiffnesses S^{μ}_{1111} ($\mu = 0, 1, 2$). For the plate thickness, we obtain the estimates

$$a_0 < h < b_0, \qquad a_1 < h < b_1, \qquad a_2 < h < b_2,$$

where $a_0 = S^0/E_{\max}$, $b_0 = S^0(1-\nu^2)/E_{\min}$, $a_1 = (8S^1/(E_{\max} - E_{\min}))^{1/2}$, $b_1 = \infty$; $a_2 = (12S^2/E_{\max})^{1/3}$, $b_2 = (12S^2/E_{\min})^{1/3}$, and $S^{\mu} = (1-\nu^2)S^{\mu}_{1111}$. Hence,

$$h_{\min} = \max\{a_0, a_1, a_2\}, \qquad h_{\max} = \min\{b_0, b_2\}. \tag{6.2}$$

7. Program and Numerical Examples. The algorithms considered above were implemented in a computer program, which consists of three main procedures: estimation of the plate thickness, solution of the CCP, and solution of the discrete CCP.

7.1. Test Problem — Redesign Problem. The redesign (alternative design) problem for laminated plates is formulated as follows. Let there be a certain design of a plate $\boldsymbol{E}^* = \{E_i,\ i = 1, \ldots, m\}$. For this design, we calculate the values of S^{μ} ($\mu = 0, 1, 2$) using formula (1.2). Then, we solve the design problem. As a result, we obtain a set of designs of the plate with the same values of S^{μ} ($\mu = 0, 1, 2$), i.e., alternative designs. If the algorithm allows one to find all designs of the plate with specified values of S^{μ} ($\mu = 0, 1, 2$), they contain the original design $\boldsymbol{E}^* = \{E_i,\ i = 1, \ldots, m\}$.

Example of the Exact Solution of the Redesign Problem. We obtain the solution of the redesign problem for the original design $\boldsymbol{E}^* = \{7, 20, 7, 20, 7, 20, 7\}$ of a seven-layered plate ($m = 7$). With a factor of 10^{10}, the values $E^*_i = 20$ correspond to steel (Young's modulus of steel is equal to $2 \cdot 10^{11}$ Pa) and the values $E^*_i = 7$ to aluminum (Young's modulus of aluminum is equal to $0.7 \cdot 10^{11}$ Pa) [11]. For this design, the stiffnesses take the values $S^0 = 12.571$, $S^1 = 0$, and $S^2 = 0.896$.

We consider the case where the plate is designed with the use of six materials with Young's moduli belonging to the set $M_6 = \{20, 7, 13, 8, 11, 9\}$. Using the algorithm described above, we obtain three designs: $\boldsymbol{E}_1 = \{7, 20, 7, 20, 7, 20, 7\}$, $\boldsymbol{E}_2 = \{11, 11, 11, 20, 13, 13, 9\}$, and $\boldsymbol{E}_3 = \{9, 13, 13, 20, 11, 11, 11\}$. For all these designs, we have $S^0 = 12.571$, $S^1 = 0$, and $S^2 = 0.896$.

The design $\boldsymbol{E}_1$ is original and designs $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ correspond to two more exact solutions. The existence of the exact redesign-problem solutions that differ from the original solution follows from the theorem on the general solution of a system of linear equations in integers [12].

Approximate Solution of the Design Problem (*Problem with the Perturbed System* $\{\boldsymbol{P}_{\gamma},\ \gamma = 1, \ldots, M+3\}$). The exact solutions of the redesign problem are given above. If the design program finds all solutions in solving the design problem in both exact and approximate formulations, all exact solutions should be among the solutions of the approximate problem. In this case, the designs that correspond to the smaller value of the perturbation parameter should be among the solutions that correspond to the larger perturbation parameter.

The problem was solved for perturbation parameters $\zeta = 0.01$ and 0.02. Numerical solutions show that, as ζ increases, new solutions appear in addition to those obtained previously.

"Dissymmetric" Solutions of the Design Problem. From the mechanical viewpoint, each design of a plate with zero nonsymmetric stiffness should correspond to the design with the same in-plane and flexural stiffnesses of the plate and "dissymmetric" arrangement of the layers. In this case, the structure can be nonsymmetric about the plane $y = 0$. The algorithm allows one to find these designs. In the solution given above, the designs $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are "dissymmetric".

Design with Allowance for Strength. As an example, we consider the design of a plate with allowance for strength. We write the strength criterion (4.2) in the form

$$f_K(\sigma^{\varepsilon}_{ij}) = (\sigma^{\varepsilon}_{ij} - \sigma^{\varepsilon}_{ij}\delta_{ij}/3)^2/(\sigma^*_K)^2 < 1,$$

where σ^*_K is the ultimate strength of the Kth material. The values of Young's moduli and ultimate strengths used in the calculations are listed in Table 1.

As was noted in Sec. 5, the strength condition is a filter for designs of plates with specified stiffnesses. As the load parameter increases, the set of designs decreases: only those designs that are able to withstand the increasing loads (strains) are retained. In the limit, we obtain the design of the strongest plate with specified stiffnesses.

TABLE 1

i	$E_i \cdot 10^{10}$, Pa	$\sigma_i^* \cdot 10^7$, Pa
1	20	30
2	7	19
3	13	30
4	8	7
5	11	10
6	9	20

Let us solve the redesign problem that corresponds to the design $\boldsymbol{E}^* = \{7, 20, 7, 20, 7, 20, 7\}$ for the set of materials $M_6 = \{20, 7, 13, 8, 11, 9\}$ with $\zeta = 0.02$. We consider cylindrical flexure of the plate of curvature ρ_{11} (ρ_{11} is the loading parameter). For $\rho_{11} < 0.07$, we obtain five designs including the original one. For $0.08 < \rho_{11} < 0.1$, two designs are obtained: $\{7, 20, 7, 20, 7, 20, 7\}$ and $\{9, 13, 13, 20, 11, 11, 11\}$. For $0.1 < \rho_{11} < 0.12$, one design is obtained: $\{9, 13, 13, 20, 11, 11, 11\}$ (design of the strongest plate). For $\rho_{11} > 0.12$, no solutions are found.

Thus, the original design $\{7, 20, 7, 20, 7, 20, 7\}$ is not the strongest one. If, in addition to the stiffnesses, it is necessary to take into account the strength properties of the plate, one should use the alternative design of the plate $\{9, 13, 13, 20, 11, 11, 11\}$ which has the same stiffnesses as the original design but can sustain stronger (by 20%) cylindrical flexure.

Design of a Plate of Unconstrained Thickness. Above, we considered some problems of design of laminated plates of specified (unit) thickness. The condition of specified plate thickness imposes a rather strong restriction on the design. In Sec. 6, we show how the plate thickness can be introduced into the number of design variables. We give the calculation results for the case where the plate thickness is a design variable. We consider the redesign problem under the condition that the alternative designs of the plate can give different thicknesses. As an original design, we use the design $\boldsymbol{E}^* = \{7, 20, 7, 20, 7, 20, 7\}$ in which the plate thickness is $h = 1$. In this case, $S^0 = 12.571$, $S^1 = 0$, and $S^2 = 0.896$. We solve the redesign problem for a thickness $h = 0.9$. For the perturbation parameter $\zeta = 0$–0.04, the problem has no solutions. For $\zeta = 0.05$, the problem is solvable and the following two designs are obtained: $\boldsymbol{E}_1 = \{13, 13, 7, 7, 20, 7, 13\}$ and $\boldsymbol{E}_2 = \{13, 12, 7, 7, 20, 8, 13\}$. For the first design, we have $S^0 = 11.429$, $S^1 = -0.02$, and $S^2 = 1.014$. The values of S^0 and S^2 differ from the original values by 9 and 13%, respectively. For the second design, we obtain $S^0 = 11.571$, $S^1 = -0.061$, and $S^2 = 1.026$. The stiffnesses S^0 and S^2 differ from the original values by 8 and 15%, respectively.

7.2. Design Problem. In the design problem, the stiffnesses are taken arbitrarily. Since the solution of the design problem of a laminated plate is equivalent to the solution of a system of integral equations of the first kind, the design problem, as a rule, has no exact solution [13]. Therefore, the design problem can be formulated and solved only approximately (this is the reason for the author's attention to the problems discussed in Sec. 3).

Given the stiffnesses, we determine the interval $[h_{\min}, h_{\max}]$ of possible values of the plate thickness using formulas (6.2) and solve the design problem for $h \in [h_{\min}, h_{\max}]$. We use a perturbation procedure that allows one to obtain both exact and approximate solutions.

Let the following stiffnesses be specified: $S^0_{1111} = 10$, $S^1_{1111} = 0$, and $S^2_{1111} = 1$. It should be noted that the quantities S^0, S^1, and S^2 in the above examples differ from S^0_{1111}, S^1_{1111}, and S^2_{1111} by an identical factor.

Formulas (6.2) yield $h_{\min} = 0.45$ and $h_{\max} = 1.3$. We solve the design problem varying the plate thickness from $h_{\min}$ to $h_{\max}$ in steps $\delta h = 0.1$. In the calculations, the perturbation parameter is taken to be $\zeta = 0.05$.

For $h = 0.5$, 0.6, and 0.7, the problem has no solutions. In particular, the CCP (6.1) has no solutions. For $h = 0.8$, 0.9, and 1.0, the problem has no solutions since discrete solutions are absent. For $h = 1.1$, 70 solutions are obtained. For $h = 1.2$, 16 designs are obtained. For $h = 1.3$, there are no solutions.

Designs with stiffnesses closest to specified values were obtained for $h = 1.1$. Hence, designs should be sought in the neighborhood of $h = 1.1$. This search was performed using the program of design of laminated plates. We give 4 out of 30 designs obtained for $h = 1.03$: $\boldsymbol{E}_1 = \{13, 8, 11, 7, 7, 13, 11\}$, $\boldsymbol{E}_2 = \{13, 8, 9, 11, 7, 9, 13\}$, $\boldsymbol{E}_3 = \{13, 9, 7, 11, 9, 8, 13\}$, and $\boldsymbol{E}_4 = \{13, 9, 8, 11, 8, 8, 13\}$. For these designs, $S^1 = 10$, $S^2 = 0$, and $S^3 = 0.944$. The stiffnesses differ from the specified values for S^3 only and the discrepancy is 6%, which is acceptable from the engineering viewpoint.

8. Case of $\boldsymbol{\nu \neq}$ const. If $\nu \neq$ const, all equations in (1.1) cannot be simultaneously reduced to the problem of the form (1.3). We consider the groups of equations

$$h^{\mu+1} \int\limits_{-1/2}^{1/2} z(y) y^{\mu}\, dy = S^{\mu}_{ijkl} \qquad (\mu = 0, 1, 2), \tag{8.1}$$

where

$$\begin{array}{llll} z(y) = z_{\rm I}(y) = E(y)/(1-\nu^2(y)) & \text{for} \quad ijkl = iiii & \text{(problem I)}, \\ z(y) = z_{\rm II}(y) = E(y)/(1+\nu(y)) & \text{for} \quad ijkl = 1212,\ 2121 & \text{(problem II)}, \\ z(y) = z_{\rm III}(y) = E(y)\nu(y)y^{\mu}/(1-\nu^2(y)) & \text{for} \quad ijkl = 1122,\ 2211 & \text{(problem III)}. \end{array} \tag{8.2}$$

For each material, the quantities E_i and ν_i are known. Therefore, one can introduce the sets $Z_{n\rm I}$, $Z_{n\rm II}$, and $Z_{n\rm III}$ such that $z_{\rm I}(y) \in Z_{n\rm I}$, $z_{\rm II}(y) \in Z_{n\rm II}$, and $z_{\rm III}(y) \in Z_{n\rm III}$. As a result, Eq. (8.1) reduces to three discrete CCPs corresponding to problems I, II, and III. We denote the general solutions of these CCPs by $\Lambda_{\rm I}$, $\Lambda_{\rm II}$, and $\Lambda_{\rm III}$. Writing the solutions of the CCPs, we replace the elastic moduli by the corresponding material numbers (see Table 1). The general solutions of the CCPs written in this form are denoted by $M_{\rm I}$, $M_{\rm II}$, and $M_{\rm III}$. Then the set of designs satisfying all three CCPs in (8.1) and (8.2) is given by

$$M_{\rm I} \cap M_{\rm II} \cap M_{\rm III}. \tag{8.3}$$

Thus, to solve the design problem for $\nu \neq$ const, it suffices to solve three CCPs (8.1), (8.2), write their solutions using the material numbers, and form set (8.3).

The difference in Poisson's ratios of materials is of significance in the only case where stiffnesses of different types [see (8.2)] are specified, for example, if the in-plane stiffness S^0_{iiii} and flexural stiffness S^2_{1212} are specified (problems I and III, respectively). If stiffnesses of one type are specified, introduction of a new variable according to (8.2) yields a problem of the form (1.3). For example, if the stiffnesses S^{μ}_{iiii} ($\mu = 0, 1, 2$) are specified (problem I), the problem is solved for the function $z_{\rm I}(y) = E(y)/(1-\nu^2(y))$. To solve this problem, one should replace the set $Z_n = \{E_i\}$ by the set $\{E_i/(1-\nu_i^2)\}$ and solve the problem in the similar manner as in the case of $\nu =$ const considered above.

9. Design of a Plate with Specified "Physical" Stiffnesses in an Arbitrary Coordinate System. The stiffnesses of the plate depend on the choice of the plane $y = 0$. At the same time, the mechanical behavior of the plate should be independent on the choice of the coordinate system. To avoid this contradiction, one should use the model of the plate written in terms of invariants [14] or specify "physical" stiffnesses.

For an arbitrary coordinate system K and a coordinate system $K(h)$ shifted by the distance h along the y axis relative to system K, we obtain the following relations for the quantities defined in (1.2) [the values of S^{μ} ($\mu = 0, 1, 2$) coincide with the values of stiffnesses with accuracy to constant factors]:

$$S^0(h) = S^0, \qquad S^1(h) = S^1 + hS^0, \qquad S^2(h) = S^2 + 2hS^1 + h^2 S^0 \tag{9.1}$$

[S^{μ} and $S^{\mu}(h)$ are calculated in the coordinate systems K and $K(h)$, respectively].

By "physical" stiffnesses, we understand stiffnesses calculated in the coordinate system in which the equality $S^1(h) = 0$ is satisfied.

From the equality $S^1(h) = 0$ in (9.1), we obtain $h = -S^1/S^0$ (equation of the "neutral"plane). For $h = -S^1/S^0$, it follows from (9.1) that

$$S^0(-S^1/S^0) = S^0, \qquad S^1(-S^1/S^0) = 0, \qquad S^2(-S^1/S^0) = S^2 - (S^1)^2/S^0. \tag{9.2}$$

By virtue of (9.2), the condition

$$S^0 = A^0, \qquad S^2 - (S^1)^2/S^0 = A^2 \tag{9.3}$$

implies that the in-plane and flexural physical stiffnesses of the plate are equal to A^0_{ijkl} and A^2_{ijkl}, respectively. In (9.3), A^0 and A^2 are the quantities that differ from the stiffnesses A^0_{ijkl} and A^2_{ijkl} by the same factors as S^{μ} differ from S^{μ}_{ijkl}; these factors depend on the type of stiffness and are described in detail in Sec. 8 [see formulas (8.2)].

Equations (9.3) are valid in an arbitrary coordinate system K. The value of S^1 is indeterminate and should be considered as a free parameter.

We estimate the values of S^1 for the coordinate system K in which the plane $y = 0$ coincides with the mid-plane of the plate:

$$S^1 = \int\limits_{-1/2}^{1/2} E(y)y\,dy \in I = \Big[-\frac{E_{\max} - E_{\min}}{8}, \frac{E_{\max} - E_{\min}}{8}\Big].$$

As a result, we obtain the problem

$$h\int\limits_{-1/2}^{1/2} E(y)\,dy = A^0_{iiii}, \quad h^2\int\limits_{-1/2}^{1/2} E(y)y\,dy = t, \quad t \in I, \quad h^3\int\limits_{-1/2}^{1/2} E(y)y^2\,dy = A^2 - \frac{t^2}{A^0}, \tag{9.4}$$

which is similar to problem (1.1) but depends on the parameter t. The parameter t is an additional design variable. The solution of problem (9.4) is constructed in the following manner. The interval I is divided into segments of equal length Δ. As a result, for each value of $t_k = -(E_{\max} - E_{\min})/8 + \Delta k$, we obtain a problem similar to those considered above.

In the examples given in Sec. **8**, it is assumed that $t = 0$, i.e., the "neutral" plane coincides with the mid-plane of the plate. It follows from (9.4) that the solutions of the design problem with the values of S^1 close to zero give a design with "physical" stiffnesses close to specified values. It is worth noting that the equality $S^1(h) = 0$ (condition that determines the "neutral" surface) is unstable with respect to perturbations. However, the stiffness characteristics are stable with respect to design perturbations.

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